DAILY UPDATE, COMPARING PID AND LQR (11/11/2019)

A PREPRINT

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Motivation

To justify in what perspective that Model predictive control (MPC), with a special format as LQR, can outperform PID controller. The most simplest case is considered in this setting, i.e., a single state system with single control input. The system dynamic is assumed to be linear, and no constraints are considered. No disturbance is considered as well. The examples are considered in continuous state formulation, and it needs to be studied how to extrapolate to the discrete space case.

LQR

The source references[1].

Consider the general case, that we want to solve the following control problem:

$$\begin{aligned}\min_{u(t)} \; J \;&=\; G(x(T)) + \int_0^T L(x(t), u(t), t)dt \\ \text{s.t. } \frac{dx}{dt} &= f(x(t), u(t), t) \\ x(0) &= x_0\end{aligned}$$

where $G(\cdot)$ is the terminal cost, J is the sum of the terminal cost and stage costs. We define a surrogate cost function $\bar{J}$ with the augmented term λ:

$$\bar{J} \;=\; G(x(T)) + \int_0^T \big(L + \lambda(f - \frac{dx}{dt})\big)dt$$

As stated in the reference, *along the trajectory, variations in J and hence $\bar{J}$ should vanish*. This follows from the fact that J is chosen to be continuous in x, u, t. Thus, the variation is expressed as

$$\delta\bar{J} = \frac{\partial G(x(T))}{\partial x}\delta x(T) + \int_0^T \big[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial u}\delta u + \lambda\frac{\partial f}{\partial x}\delta x + \lambda\frac{\partial f}{\partial u}\delta u - \lambda\delta\frac{dx}{dt}\big]dt$$

The last term above can be evaluated using integration by parts as

$$-\int_0^T \lambda\delta\frac{dx}{dt}dt = -\lambda(T)\delta x(T) + \lambda(0)\delta x(0) + \int_0^T \frac{d\lambda}{dt}\delta x dt$$

and bring in back we have

$$\begin{aligned}\delta\bar{J} \;&= \frac{\partial G(x(T))}{\partial x}\delta x(T) + \int_0^T \big(\frac{\partial L}{\partial u} + \lambda\frac{\partial f}{\partial u}\big)\delta u dt + \\ &\int_0^T \big(\frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} + \frac{d\lambda}{dt}\big)\delta x dt - \lambda(T)\delta x(T) + \lambda(0)\delta x(0)\end{aligned}$$

[1] https://ocw.mit.edu/courses/mechanical-engineering/2-154-maneuvering-and-control-of-surface-and-underwater-vehi
lecture-notes/lec19.pdf

The last term is zero, since *we cannot vary the initial condition of the state by changing something later in time* - it is a fixed value. To make the variation converges to zero the following conditions must be satisfied:

$$\begin{aligned} \frac{\partial L}{\partial u} + \lambda \frac{\partial f}{\partial u} &= 0 \\ \frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} + \frac{d\lambda}{dt} &= 0 \\ \frac{\partial G(x(T))}{\partial x} - \lambda(T) &= 0 \end{aligned}$$

Thus, we can derive the LQR from the general steps above. For example, for LQR problem without terminal cost, i.e., $G(x(T)) = 0$, and

$$L = \frac{1}{2}Qx^2 + \frac{1}{2}Ru^2$$

where $Q > 0, R > 0$. In the case of linear dynamic system, $\frac{dx}{dt} = Ax + Bu$, we have

$$\frac{\partial L}{\partial x} = Qx, \quad \frac{\partial L}{\partial u} = Ru, \quad \frac{\partial f}{\partial x} = A, \quad \frac{\partial f}{\partial u} = B$$

using the zero variation condition we have

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ x(0) &= x_0 \\ \frac{d\lambda}{dt} &= -Qx - A\lambda \\ \lambda(T) &= 0 \\ Ru + B\lambda &= 0 \end{aligned}$$

$\lambda = Px$ is guessed initially. Then we obtain

$$PAx + APx + Qx - P^2B^2x/R + \frac{dP}{dt} = 0$$

which is the matrix Riccati equation. The steady-state solution is given

$$PA + AP + Q - P^2B^2/R = 0$$

and finally we get the fixed control policy

$$u = -BPx/R$$

One step further, for single state and single input system, we have

$$P = \frac{AR + \sqrt{A^2R^2 + B^2QR}}{B^2}$$

so that we have

$$u = -\frac{A + \sqrt{A^2 + B^2Q/R}}{B}x = k_u x$$

Finally, the dynamic system becomes

$$\frac{dx}{dt} = (A + Bk_u)x$$

and the solution is

$$x(t) = x_0 \exp((A + Bk_u)t)$$

and eventually

$$x(t) = x_0 \exp(-\sqrt{A^2 + B^2Q/R}t)$$

PID

reference[2]. First we consider the commonly used PI controller. Assume x_e, u_e are the reference state and input, then the error term is $x - x_e$. Define

$$e = x_e - x$$

then the system dynamic is equivalent to

$$-\frac{de}{dt} = -Ae + Bu$$

Given the control policy of PI controller is

$$u = k_p\, e + k_i \int_0^T e(\tau) d\tau$$

First differentiate the system dynamic and control policy, we have

$$-\frac{d^2e}{dt^2} = -A\frac{de}{dt} + B\frac{du}{dt}$$

and

$$\frac{du}{dt} = k_p\, \frac{de}{dt} + k_i e$$

then bring it (du/dt) back we have

$$\frac{d^2e}{dt^2} + (-A + Bk_p)\frac{de}{dt} + Bk_i e = 0$$

Obviously, PI controller leads to second order dynamic system, of the format

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = 0$$

As indicated in the reference, a desired property for the controller is to respond to changes smoothly without oscillations, which is equivalent to set $\zeta = 1$, the critical damping. Also, the choice of ω_0 is a compromise between response speed and control actions. Since

$$k_p = \frac{A + 2\zeta\omega_0}{B}, \qquad k_i = \frac{\omega_0^2}{B}$$

it is equivalent to say that the choice of k_p, k_i determine the natural frequency and damping factor of the system. Finally, the solution of the system is

$$x(t) = x(0)(1 - \omega_0 t)\exp(-\omega_0 t)$$

which leads to

$$x(t) = x_0(1 - \frac{Bk_p - A}{2}t)\exp(\frac{A - Bk_p}{2}t)$$

[2]Chapter 10, Murray