Useful Math Tools for ML Proofs

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Introduction

Inequality

Cauchy-Schwarz inequality

For any vector $\boldsymbol{u}$ and $\boldsymbol{v}$ of an inner product space, it is true that

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle^2 \leq \langle \boldsymbol{u}, \boldsymbol{u} \rangle \cdot \langle \boldsymbol{v}, \boldsymbol{v} \rangle$$

Jensen's inequality

It relates the value of a convex function of an integral of the convex function. in the context of convex function, for convex function $f(\cdot)$ and $t \in [0, 1]$:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

in the context of probability theory, it is generally stated in the following form: it X is a random variable and φ is convex function, then

$$\varphi\big(\mathrm{E}(X)\big) \leq \mathrm{E}[\varphi(X)]$$

the difference between the two sides of the inequality is called the Jensen gap.

Lipschitz continuity

A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is called Lipschitz continuous if there exists a postive real constant K such that, for all real x_1 and x_2,

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$$

Markov's inequality

If X is a non-negative random variable and $a > 0$, then the probability that X is at least a is at most the expectation of X divided by a:

$$\mathrm{P}(X \geq a) \leq \frac{\mathrm{E}(X)}{a}$$

Approximation

Taylor's Expansion

$$f(x) \approx f(a) + \frac{f'(a)}{1!} + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$